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ON CRITICAL VALUES OF ADJOINT L-FUNCTIONS FOR GSp(4)

ATSUSHI ICHINO

1. Introduction

Let $f \in S_k(\mathrm{SL}(2,\mathbf{Z}))$ be a normalized Hecke eigenform and $\pi = \otimes_v \pi_v$ the irreducible cuspidal automorphic representation of $\mathrm{GL}(2,\mathbf{A_Q})$ determined by f. Then the result of Rankin [12] says that

$$L(1,\pi,\mathrm{Ad}) = C_\infty \langle f, f \rangle,$$

where $\mathrm{Ad} : \mathrm{GL}(2,\mathbf{C}) \to \mathrm{GL}(3,\mathbf{C})$ is the adjoint representation, $C_\infty = 2^k$ is a constant which depends only on π_∞, and

$$\langle f, f \rangle = \int_{\mathrm{SL}(2,\mathbf{Z})\backslash\mathfrak{H}} |f(\tau)|^2 \, \mathrm{Im}(\tau)^{k-2} \, d\tau$$

is the Petersson norm of f. This formula was generalized to the case of $\mathrm{GL}(n)$ by Jacquet, Piatetski-Shapiro, and Shalika [6]. In this note, we give an analogue for GSp(4).

2. Deligne's conjecture [3]

We first give some speculation about the transcendental part of critical values of adjoint L-functions for GSp(4). Let f_{hol} be a Siegel cusp form of degree 2 and of weight k with respect to $\mathrm{Sp}(4,\mathbf{Z})$. We assume that f_{hol} is a Hecke eigenform and is not a Saito-Kurokawa lift. Let π_{hol} be the irreducible cuspidal automorphic representation of $\mathrm{GSp}(4,\mathbf{A_Q})$ determined by f_{hol}. By Arthur's conjecture [1], there would exist an irreducible generic cuspidal automorphic representation π_{gen} of $\mathrm{GSp}(4,\mathbf{A_Q})$ such that $\Pi = \{\pi_{\mathrm{hol}}, \pi_{\mathrm{gen}}\}$ is an L-packet. Namely,

$$L(s,\pi_{\mathrm{hol}},r) = L(s,\pi_{\mathrm{gen}},r)$$

for any finite dimensional representation r of $\mathrm{GSp}(4,\mathbf{C})$. Let M be the hypothetical motive attached to the spinor L-function of f_{hol}. Then M would be of rank 4 and of pure weight $2k-3$. Moreover, the Hodge decomposition

$$H_{\mathrm{DR}}(M) \otimes \mathbf{C} \cong H^{2k-3,0} \oplus H^{k-1,k-2} \oplus H^{k-2,k-1} \oplus H^{0,2k-3}$$

would have a basis

$$\left\{ f_{\mathrm{hol}}, f_{\mathrm{gen}}, \overline{f_{\mathrm{gen}}}, \overline{f_{\mathrm{hol}}} \right\}.$$

Here f_{gen} is an element of π_{gen}. By Yoshida's formula [13, (4.15)], we have

$$c^+(\mathrm{Sym}^2(M)) = \left(2\pi\sqrt{-1}\right)^{12-6k} c^+(M) c^-(M) \langle f_{\mathrm{hol}}, f_{\mathrm{hol}} \rangle,$$

where $c^+(\mathrm{Sym}^2(M))$ is Deligne's period of $\mathrm{Sym}^2(M)$, etc. Moreover, the relative trace formula of Furusawa and Shalika [4] suggests that the equality

$$\frac{|B_D(1)|^2}{\langle f_{\mathrm{hol}}, f_{\mathrm{hol}}\rangle} = L\left(\tfrac{1}{2}, \Pi\right) L\left(\tfrac{1}{2}, \Pi \otimes \chi_D\right) \frac{|W(1)|^2}{\langle f_{\mathrm{gen}}, f_{\mathrm{gen}}\rangle}$$

should hold up to an elementary constant. Here $D < 0$ is a fundamental discriminant, χ_D is the Dirichlet character associated to $\mathbf{Q}\left(\sqrt{D}\right)/\mathbf{Q}$, B_D is the D-th Bessel function of f_{hol}, and W is the Whittaker function of f_{gen}. This leads to speculation that

$$c^+(\mathrm{Sym}^2(M)) \stackrel{?}{=} \langle f_{\mathrm{gen}}, f_{\mathrm{gen}}\rangle.$$

3. Result

We now give a precise description of our result. Let

$$\mathrm{GSp}(4) = \left\{ g \in \mathrm{GL}(4) \;\middle|\; g \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix} {}^t g = \nu(g) \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix}, \ \nu(g) \in \mathbf{G}_m \right\}$$

be the symplectic similitude group in four variables. Let $\pi = \otimes_v \pi_v$ be an irreducible generic cuspidal automorphic representation of $\mathrm{GSp}(4, \mathbf{A}_\mathbf{Q})$ with trivial central character. We assume that

- π_p is unramified for all primes p,
- $\pi_\infty|_{\mathrm{Sp}(4,\mathbf{R})} = D_{(\lambda_1,\lambda_2)} \oplus D_{(-\lambda_2,-\lambda_1)}$ with $1 - \lambda_1 \le \lambda_2 \le 0$.

Here $D_{(\lambda_1,\lambda_2)}$ is the (limit of) discrete series representation of $\mathrm{Sp}(4, \mathbf{R})$ with Blattner parameter (λ_1, λ_2). By [2], π has a functorial lift Π to $\mathrm{GL}(4, \mathbf{A}_\mathbf{Q})$. We assume that Π is cuspidal.

We consider a non-zero element $f = \otimes_v f_v \in \pi$ satisfying the following conditions:

- f_p is $\mathrm{GSp}(4, \mathbf{Z}_p)$-invariant for all primes p,
- f_∞ is the lowest weight vector of the minimal U(2)-type of $D_{(-\lambda_2,-\lambda_1)}$.

Note that f is unique up to scalars. We may normalize f so that $W(1) = 1$, where W is the Whittaker function of f. Let

$$\langle f, f\rangle = \int_{\mathbf{A}_\mathbf{Q}^\times \mathrm{GSp}(4,\mathbf{Q})\backslash \mathrm{GSp}(4,\mathbf{A}_\mathbf{Q})} |f(g)|^2 \, dg$$

be the Petersson norm of f, where dg is the Tamagawa measure on $\mathrm{GSp}(4, \mathbf{A}_\mathbf{Q})$.

Our main result is as follows.

Theorem 3.1 ([5]). *There exists a constant $C_\infty \in \mathbf{C}^\times$ which depends only on π_∞ such that*

$$L(1, \pi, \mathrm{Ad}) = C_\infty \langle f, f\rangle.$$

Here $\mathrm{Ad} : \mathrm{GSp}(4, \mathbf{C}) \to \mathrm{GL}(10, \mathbf{C})$ *is the adjoint representation.*

4. Proof

We use the following three ingredients:

- the integral representation of $L(s, \pi, \mathrm{St})$,
- the integral representation of $L(s, \pi \times \pi^\vee) = \zeta(s) L(s, \pi, \mathrm{St}) L(s, \pi, \mathrm{Ad})$,
- the Siegel-Weil formula.

Let $H = \mathrm{GSp}(8)$ and

$$G = \{(g_1, g_2) \in \mathrm{GSp}(4) \times \mathrm{GSp}(4) \mid \nu(g_1) = \nu(g_2)\}.$$

We identify G with its image under the embedding

$$G \longrightarrow H.$$

$$\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \longmapsto \begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & -b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & -c_2 & 0 & d_2 \end{pmatrix}$$

For an automorphic form φ on $H(\mathbf{A_Q})$, let

$$\langle \varphi|_G, \bar{f} \otimes f \rangle = \int_{Z_H(\mathbf{A_Q}) G(\mathbf{Q}) \backslash G(\mathbf{A_Q})} \varphi((g_1, g_2)) f(g_1) \overline{f(g_2)} \, dg_1 \, dg_2.$$

Let

$$P = \left\{ \begin{pmatrix} a & * \\ 0 & \nu {}^t a^{-1} \end{pmatrix} \in H \;\middle|\; a \in \mathrm{GL}(4),\ \nu \in \mathbf{G}_m \right\}$$

be the Siegel parabolic subgroup of H. Let $F = \otimes_v F_v$ be a holomorphic section of $\mathrm{Ind}_{P(\mathbf{A_Q})}^{H(\mathbf{A_Q})} \left(\delta_P^{s/5}\right)$, where δ_P is the modulus character of $P(\mathbf{A_Q})$. Let $E(s, F)$ be the Siegel Eisenstein series attached to F.

Theorem 4.1 (Piatetski-shapiro and Rallis [11]). *We have*

$$\langle E(s, F)|_G, \bar{f} \otimes f \rangle = \langle f, f \rangle d_P^S(s)^{-1} L^S\left(s + \tfrac{1}{2}, \pi, \mathrm{St}\right) \prod_{v \in S} Z_v(s, \phi_v, F_v).$$

Here $d_P^S(s) = \zeta^S\left(s + \frac{5}{2}\right) \zeta^S(2s+1) \zeta^S(2s+3)$, *$\phi_v$ is the matrix coefficient of π_v associated to f_v such that $\phi_v(1) = 1$, and $Z_v(s, \phi_v, F_v)$ is the local zeta integral.*

Let

$$Q = \left\{ \begin{pmatrix} a & * & * & * \\ 0 & * & * & * \\ 0 & 0 & \nu {}^t a^{-1} & 0 \\ 0 & * & * & * \end{pmatrix} \in H \;\middle|\; a \in \mathrm{GL}(3),\ \nu \in \mathbf{G}_m \right\}$$

be a maximal parabolic subgroup of H. Let $\mathcal{F} = \otimes_v \mathcal{F}_v$ be a holomorphic section of $\mathrm{Ind}_{Q(\mathbf{A_Q})}^{H(\mathbf{A_Q})} \left(\delta_Q^{s/6}\right)$, where δ_Q is the modulus character of $Q(\mathbf{A_Q})$. Let $\mathcal{E}(s, \mathcal{F})$ be the Eisenstein series attached to $\mathcal{F}$.

Theorem 4.2 (Jiang [7]). *We have*

$$\langle \mathcal{E}(s, \mathcal{F})|_G, \bar{f} \otimes f \rangle = d_Q^S(s)^{-1} L^S\left(\tfrac{s+1}{2}, \pi \times \pi^\vee\right) \prod_{v \in S} \mathcal{Z}_v(s, W_v, \mathcal{F}_v).$$

Here $d_Q^S(s) = \zeta^S(s+1)\zeta^S(s+2)\zeta^S(s+3)\zeta^S(2s+2)$*,* W_v *is the Whittaker function of* π_v *associated to* f_v *such that* $W_v(1) = 1$*, and* $\mathcal{Z}_v(s, W_v, \mathcal{F}_v)$ *is the local zeta integral.*

To compare these two integral representations, we use the Siegel-Weil formula. Recall the analytic behavior of the Eisenstein series:

- $E(s, F)$ has at most a simple pole at $s = \frac{1}{2}$ (Kudla and Rallis [10]),
- $\mathcal{E}(s, \mathcal{F})$ has at most a double pole at $s = 1$ (Jiang [7]).

On the other hand, since Π is cuspidal,

- $L^S\left(s + \frac{1}{2}, \pi, \mathrm{St}\right)$ is holomorphic and non-zero at $s = \frac{1}{2}$,
- $L^S\left(\frac{s+1}{2}, \pi \times \pi^\vee\right)$ has a simple pole at $s = 1$.

Hence the first terms in the Laurent expansions of the Eisenstein series do not contribute to special values. This means that we must compare the second terms.

Proposition 4.3. *There exist* F *and* $\mathcal{F}$ *which are* $H(\hat{\mathbf{Z}})$*-invariant and which satisfies the following:*

- *For*

$$\varphi = \mathrm{Res}_{s=1}\, \mathcal{E}(s, \mathcal{F}) - \zeta(4)^{-1}\, \mathrm{CT}_{s=\frac{1}{2}}\, E(s, F),$$

we have

$$\langle \varphi|_G, \bar{f} \otimes f \rangle = 0.$$

- $Z_\infty(s, \phi_\infty, F_\infty)$ *is holomorphic and non-zero at* $s = \frac{1}{2}$.
- $\mathcal{Z}_\infty(s, W_\infty, \mathcal{F}_\infty)$ *is holomorphic and non-zero at* $s = 1$.

The proof of this proposition is based on the regularized Siegel-Weil formula of Kudla and Rallis [10], Kudla [9], and Jiang [8].

Now it is easy to check that

$$\frac{L(1, \pi, \mathrm{Ad})}{\langle f, f \rangle} = C'_\infty \frac{Z_\infty\left(\frac{1}{2}, \phi_\infty, F_\infty\right)}{\mathcal{Z}_\infty(1, W_\infty, \mathcal{F}_\infty)},$$

where

$$C'_\infty = 2^{-1}\zeta_\infty(4)^{-1} L_\infty(1, \pi_\infty, \mathrm{Ad}) \in \mathbf{C}^\times.$$

Since ϕ_∞ and W_∞ depend only on π_∞, the right-hand side depends only on π_∞, F_∞, and $\mathcal{F}_\infty$. However, the left-hand side is independent of F_∞ and $\mathcal{F}_\infty$. This completes the proof of Theorem 3.1.

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DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA CITY UNIVERSITY, 3-3-138 SUGIMOTO, SUMIYOSHI-KU, OSAKA 558-8585, JAPAN

E-mail address: `ichino@sci.osaka-cu.ac.jp`